AROUND PERELMAN'S PROOF OF THE POINCARÉ CONJECTURE

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Abstract. Certain life principles of Perelman may look unusual, as it often happens with outstanding people. But his rejection of the Fields medal seems natural in the context of various activities followed his breakthrough in mathematics.

Cet animal est très méchant,
quand on l'attaque, il se défend.

Folklore

After 3 years of thorough inspection, Perelman's proof of the Poincaré Conjecture was finally recognized by the consensus of experts. Three groups of researchers not only verified its details, but also published their own expositions, clarifying the proofs in several hundreds of pages, on the level "available to graduate students". The solution of the Poincaré Conjecture was the central event discussed in the International Congress of Mathematicians in Madrid, in August 2006 (ICM-2006), which gave to it a flavor of a historical congress.

Additional attention to the personality of Perelman was attracted by his rejection of the Fields medal. Some other of his decisions are also non-conventional and tradition-breaking, for example Perelman's refusal to submit his works to journals. Many people find these decisions strange, but I find them logical and would rather criticize certain bad traditions in the modern organization of science. One such tradition is a kind of tolerance to unfair credit for mathematical results. Another big problem is the journals, which are often transforming into certain clubs closed for aliens, or into profitable businesses, enjoying the free work of many mathematicians. Sometimes the role of the "organizers of science" in mathematics also raises questions.

Some biographical information about Perelman. Grigory (Grigorij, or Gregory) Perelman was born in 1966, June 13, in Leningrad (now St.Petersburg). He used to sign his papers with the nickname "Grisha", which gives the feeling of an informal and friendly style.

Grisha graduated from the high school No 239, which was very famous in Leningrad and perhaps the best one in the time of Perelman. Several such elite schools in Russia (two more famous schools, No 30 and No 45 in Leningrad and several similar schools in Moscow) were growing young talents, whose abilities were often reflected in their success on the International Mathematical Olympiads (IMO). Participating in IMO-1982 in Budapest, Perelman got a gold medal with the perfect score (42 out

of 42). Grisha graduated from the Leningrad State University and then continued his Ph.D. studies in Geometry. His formal supervisor was A.D.Aleksandrov (who did not look very active by that time, because of his age), and the actual one was Yu.D.Burago. The policy the Steklov Mathematical Institute was to restrict Jews from getting Ph.D.'s, and even such a talent as Perelman could become a victim. By writing Aleksandrov as a supervisor, Burago aimed to avoid such a possibility (Aleksandrov had a title of *academic*, and used to be the rector of the Leningrad University many years ago, so not to accept his student for Ph.D. studies would become a scandal).

After defending his Ph.D. in 1990, Grisha has shown himself as one of the top experts in the *Aleksandrov geometry*, after he solved in 1990-91 some hard problems in this subject. As such an expert, Grisha was invited to the USA, where he was working for several years and obtained also a reputation of an "incredibly brilliant phenomenon" in Riemannian geometry. The most famous of Perelman's achievements during his work in the USA was *the Soul Conjecture* formulated in 1972 by Cheeger and Gromoll, to which Grisha found a remarkably short solution in 1994. I was told by Burago that he however considers Perelman's previous results even more fundamental and important for geometry than the Soul Conjecture (and their proofs were much more complicated). In 1992 Grisha spent one semester in the New York University and one more semester in Stony Brook University. In 1993–1995 he had a fellowship position in the University of California at Berkeley.

The sign of a formal recognition of his achievements, was the invited lecture at the ICM-1994 in Zürich, in the Geometry section. He was given some awards, and rejected some of them, for example, in 1995, an award by the European Math Society given to ten young mathematicians. He had an opportunity to get a job in such elite institutions as Stanford and Princeton Universities, Institute for Advanced Studies, but did not show interest in the offers. As I was told by Ya.Eliashberg, the selection committee in the Stanford University decided that they ccould not handle such a strange person, after Perelman refused to submit his Curriculum Vitae there. According to the article of S.Nasar in "New Yorker", Grisha's idea was *if they do know my works, then they don't need to ask my CV; if they do not know, then there is no need to work there.* Finally, in 1995 Grisha returned to Russia.

Some of his colleagues abroad reported that they had an impression that he went into a shadow and was not showing signs of an active research life, if not to count rare e-mail messages. Perhaps, it looked contrasting with his previous numerous results and publications. The interested people were curiously asking Perelman's colleagues in the Steklov Institute: what is he doing ? However, nobody had the slightest idea. One can guess that Grisha returned to home in St.Petersburg to concentrate on the Poincaré conjecture.

If so, it cannot look surprising that Grisha was avoiding extra activities and paperwork required for promotional purposes. Getting a senior academic position in Russia requires defending the second, "higher" doctoral degree (habilitation), which assumes perhaps more social efforts than real research. Grisha perhaps did not want to spend substantial time playing these career games, and was not even publishing papers for several years, as he was working on the Poincare Conjecture. As I was told by Perelman's boss in the Steklov Institute, Grisha was refusing to provide research plans and annual reports on his work. Perhaps, this kind of problems in Perelman's relations with the administration were collecting and he finally quit from the Institute saying that he is "leaving from the professional

mathematics".

In my opinion, a mathematician of such a level deserves very special treatment. For example, it is ridiculous that Perelman still remains outside the circle of *academics*, who are supposed to be the mathematical elite in Russia. I can imagine some formal reasons for this, but whatever they are, it demonstrates a deep difference between the real scientists and the career-makers in the academic world (and I am afraid, not only in the Russian Academy of Science).

Ricci flow program of Hamilton-Yau. It is wrong to think that Perelman created his theory from scratch. His breakthrough was prepared by the efforts of several remarkable mathematicians.

The first such mathematical giant was W.Thurston, who formulated *the Geometrization Conjecture* on the structure of 3-dimensional manifolds. It says that any 3-manifold can be cut along spheres and incompressible tori into nicely homogeneous, so called *geometric* pieces, according to 8 existing *geometries*. The Poincaré Conjecture turns out to be a special case of the Geometrization Conjecture. The reason is that a homotopy sphere, Σ, does not contain incompressible surfaces. In such a case, the Geometrization Conjecture means that Σ splits into a connected sum of geometric pieces. However, this splitting cannot be essential (without S^3-summands), since there is no geometric homotopy balls topologically different from D^3.

Thurston himself made some progress in proving his conjecture, however not in the most interesting case, related to the Poincaré Conjecture.

The same year, 1982, as Thurston received a Fields medal for his remarkable contribution in theory of surfaces and 3-manifolds, R.Hamilton published a paper in which he analyzed *the Ricci Flow*, an evolution equation for a time-dependent Riemannian metric, $g = g(t)$, on a manifold M

$$\frac{dg}{dt} = -2Ric(g)$$

Here $Ric(g)$ is the Ricci curvature tensor, which depends on some partial derivatives of the tensor g. The Ricci flow equation is of parabolic type, like the heat equation (although it is not linear and involves tensors rather than functions). The evolution is supposed to improve the irregularities of the initial metric g, making it more homogeneous, like heat in space becomes more homogeneously distributed as time passes.

Recall that one of the insights of A.Einstein is his interpretation of gravitation as a Riemannian metric. Some version of the Ricci flow equation was considered by Einstein in this context, and the *constant solutions* to such an equation, that is metrics g proportional to $Ric(g)$, are now known as *Einstein metrics*. They play an important role in Riemannian geometry, as well as *Kähler-Einstein metrics* are important in the complex geometry. S.Yau received a Fields medal, the same time as Thurston, in 1982, for proving the *Calabi Conjecture* about existence (under certain assumptions) of the Kähler-Einstein metrics.

As Hamilton remarked, it was Yau who explained to him how analysis of the Ricci flow may be related to solution of the Geometrization, and thus, of the Poincaré Conjecture. In the 80s and beginning of the 90s, Hamilton made several essential steps in this direction. He has shown that if the Ricci flow behaves nicely, then its limit solves the problem of geometrization. But he was stuck with possible

singularities that may appear if the Ricci Flow does not behave nicely. This problem required the insight and the incredible technical power of Perelman.

To clarify the context in which Perelman's solution appeared, I should say that, unlike the Geometrization conjecture which was known perhaps to all working in low-dimensional topology, the Ricci Flow program of Hamilton-Yau was not well known outside a restricted group of geometric analysts. It did not seem at all that this program might be completed soon, because the subject was technically hard and progress was slow. Other projects of proving the Poincaré Conjecture looked not less promising. At least, I have not heard about Hamilton's program before Perelman's work, although the Poincaré Conjecture is the central question in my special field of interest, and I was always interested in any serious development around it.

History of Perelman's proof. During his stay in Berkeley, Perelman learned about the Ricci flow project from Hamilton. At some point (may be after years), he realized that the theory of Aleksandrov spaces, in which he was one of the top experts, could be useful for analysis of the singularities of the Ricci flow.

He concentrated on this subject after return to Russia, and wrote a letter to Hamilton in 1996 describing his ideas and proposing to work together. Hamilton did not show interest in Perelman's offer and Grisha decided to work on his own.

The first of his papers, "The entropy formula for the Ricci flow and its geometric applications", was submitted to the web preprint-server "ArXive" on 11.11.2002. Grisha sent letters to Hamilton, Yau, Tian, Morgan (and may be some others) to attract attention to his goals, since it was not obvious from the title which kind of a bomb he was exploding. It is interesting that Grisha did not even make an explicit claim that he is proving the Poincaré and the Geometrization Conjecture, it was hidden somewhere between the lines. This really caused some misunderstanding for non-experts: one of my senior friends-mathematicians was expressing his surprise: how can everybody speak about Perelman's proof of the Poincaré Conjecture, if its claim cannot be found explicitly in his papers ? It seems that Grisha wanted to show modesty and patience, letting other people verify his arguments, before making any claim out loudly. May be Grisha did not want to be in the situation of Andrew Wiles, who after a triumphal announcement of the solution to the "Fermat Theorem", faced an essential gap found in his arguments (although with the help of colleagues, Wiles fortunately managed to repair his proof in a few months).

Anyway, Perelman's high reputation required news from him to be considered very seriously, even though many details were not clear in the beginning. Grisha accepted an invitation to USA, to clarify the details to the experts there. To complete his first publication, Grisha submitted to the ArXive his second paper "Ricci Flow with surgery on 3-manifolds", on March 10, 2003, and then visited USA giving a series of lectures: in Princeton, MIT, Columbia University, Stony Brook. This visit abroad, in the Spring 2003, seems to be the only one since 1995. He kept declining all other offers to visit, even to the conferences devoted to his own publications, perhaps, considering such visits unnecessary. According to some sources, Perelman's point of view was that he *should not try to interfere in the process of verification of his proof, staying aside and letting people check themselves the text which was actually presented.*

The third and the last of his papers " Finite extinction time for the solutions to the Ricci flow on certain three-manifolds" appeared in the ArXive in July 17,

2003. It contained a shortcut to the proof of the Poincaré Conjecture, avoiding some technicalities of his approach to the more general result, Geometrization Conjecture given in the previous two papers.

None of these three papers was submitted to a journal, which is one of Perelman's non-traditional decisions. From common sense, an important work should be published in a journal, which means that some expert will check if it is correct and readable. Another reason in the past was to make the paper available for readers via libraries, but nowadays after ArXive appeared, this argument probably lost its importance. In the case of Perelman, journal reviewing looks also meaningless. His work was reviewed by all the top experts, and more carefully and longer than any journal could expect. It would become some strange lottery to choose between journals and let some particular reviewers take decisions about the work of such importance. One more reason for journal publication could be an illusion of some protection of the authors rights. But Perelman emphasized that he cares about mathematics, but not about the copyright, prizes and other social games around it.

Inspection and recognition of Perelman's proof. In June 2003 John Lott (University of Michigan) and Bruce Kleiner (now in Yale University) opened a webpage where they were reporting about the progress in understanding Perelman's proofs. There were several conferences: in the end of 2003, a workshop in Berkeley devoted to the first Perelman paper; in the summer 2004, a workshop in Princeton on his second paper; a summer school by the Clay Institute in 2005. One can also mention many other activities. For example, a Bourbaki Lecture by Gérard Besson, in June 2005 (see Artérisque series, September 2006), and a series of lectures at Harvard University by Xi-Ping Zhu, during six month in the fall of 2005.

This inspection made the experts agree that Perelman's proof has no real gaps. Some technical details were of course omitted, which is natural in a presentation of a serious theory designed for experts only in 60 pages totally. Some of the experts, after verifying Perelman's theory, produced their own more detailed expositions of this theory. The most important and well-known are three publications that appeared in 2006: by Kleiner and Lott, " Note on Perelman's paper", submitted to the web on May 25; by Cao and Zhu, 330 pages published in the June issue of "Asian journal of mathematics" (but unfortunately unavailable to mathematicians through the web); and by Morgan and Tian, "Ricci Flow and the Poincaré Conjecture", 470 pages, appeared on the web on July 25 (a book to be published in the beginning of 2007).

These expositions differ in character and each of them is an important contribution clarifying to us Perelman's theory. The one of Kleiner-Lott as well as of Cao-Zhu are based principally on the first two papers of Perelman and follow his proof of the Geometrization Conjecture, with the Poincaré Conjecture as its particular case. The exposition by Morgan-Tian is based mainly on the third of Perelman's papers, providing a shortcut for proving the Poincaré Conjecture, without treating the more general Geometrization Conjecture.

John Lott commented: *"Perelman's articles are remarkably carefully written if one takes into account how much new ground he breaks in a relatively few number of pages. However, they are not written in such a way that one can just sit down and quickly decide whether his arguments are complete."* John Morgan (Columbia University) remarked that Perelman omits certain technical points that turned out to be standard, but rather tricky to work out in detail.

Controversial claims of credits by Cao and Zhu. The value of the work of Cao-Zhu was undermined by strange claims of credit, motivated by lack of details in Perelman's papers. These authors made some modifications in the original arguments to avoid using the theory of Aleksandrov spaces (although the reasons for it are not clear). The abstract of their paper is as follows:

"In this paper, we give a complete proof of the Poincaré and the geometrization conjectures. This work depends on the accumulative works of many geometric analysts in the past thirty years. This proof should be considered as the crowning achievement of the Hamilton-Perelman theory of Ricci flow".

In his speech on the International String Theory Conference which was held in Beijing, in June 2006, Yau claimed

"In Perelman's work many key ideas of the proof are sketched or outlined, but complete details of the proof are often missing. The recent paper of Cao-Zhu gives the first complete and detailed account of the proof of the Poincaré and the geometrization conjectures."

A propaganda campaign organized in the Chinese mass media could probably completely disorient the public and some part of the mathematical community in China. Let me give an example from a Chinese newspaper, with a typical heading *"Chinese mathematicians solve global puzzle"*, which announces: *Two Chinese mathematicians, Zhu Xiping and Cao Huaidong, have put the final pieces together in the solution to the puzzle that has perplexed scientist around the globe for more than a century.*. Another example: *A Columbia professor Richard Hamilton and a Russian mathematician Grigori Perelman have laid foundation on the latest endeavors made by the two Chinese."* From a Chinese news agency report about ICM-2006: *Based on several manus posted on the internet, Russian mathematician Perelman claimed that he completed this program. A complete exposition was recently given by Cao and Zhu.* Comments of some Chinese mathematicians sound like this: *Perelman raised guidelines for proving this conjecture but not specifically pointed out how to unravel the puzzle. Guidelines are totally different to complete proof of theories*

Unlike Cao and Zhu, the other experts inspecting Perelman's proof were respectful to him and modestly refrained from claiming credit for themselves. In the introduction by Kleiner and Lott we can read: *" ... Perelman's proofs are concise and, at times, sketchy. The purpose of these notes is to provide the details that are missing ... "*

In the interview during ICM-2006 in Madrid, when J.Morgan was asked about the claims of Cao-Zhu he replied, *"Perelman already did it, and what he did was complete and correct. I do not see anything that [these authors] made differently."*

Perelman himself remarked about Cao-Zhu: *"It is not clear to me what new contribution did they make. Apparently, Zhu did not quite understand the argument and revoked it."*

Many people blame Yau, who has an enormous influence in China, as a person responsible for the claims of Cao and Zhu and for the campaign that followed their publication. Yau was the doctoral advisor of Cao, and he organized a study group on Perelman's proof, supported by Clay's Institute and other funds. Yau is also the editor-in-chief of the Asian Journal of Mathematics, which published the paper of Cao-Zhu in an unusually short period of time and, according to some complaints, without proper refereing.

More details on Yau's story can be found in the article "Manifold destiny" by

Sylvia Nasar (the author of "Beautiful mind") and David Grubner, published in "New Yorker".

According to this publication, Perelman remarked, concerning these claims of credit, *"I can't say I'm outraged. Other people do worse. Of course, there are many mathematician who are more or less honest. But almost all of them are conformists. They are more or less honest, but they tolerate those who are not honest."* Perhaps, he wanted to say that he finds more important not the exceptional cases of explicit mishandling, but the quite common reaction to it from the majority, from those who do not care and tolerate such things, still considering themselves personally honest.

After the publication in "New Yorker", Yau applied to a court against the newspaper and its authors, saying that he had never said the phrase attributed to him: *"50% of the credit for the solution of the Poincare Conjecture should go to Cao-Zhu, 25% to Perelman, and 30% to Hamilton."* Yau also claims that the campaign against him is secretly directed by another prominent Chinese mathematician, his former student, and now a sort of rival. Hamilton published a letter defending Yau against the anger of many other mathematicians.

Awards and manipulation by public's opinion. The Fields medal was awarded to Perelman by the International Math Union (IMU) with the following formulation *" for his contribution to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow".*

It leaves an impression that the IMU is playing with words: everybody in ICM-2006 was understanding that the main reason to award Perelman the Fields medal was for his solution of the Poincare conjecture, but instead of it we see only some general "revolutionary insights". From the goal (Poincare Conjecture) our attention was switched to the tool (Ricci flow). A contribution to topology is not mentioned at all, although the greatest topological problem was solved. It is hard to believe that the IMU did not work carefully enough on this kind of formulation.

At the first moment, it sounds in accord with Yau's point of view, that the solution of the Poincaré conjecture is a great triumph of Chinese mathematics, although Perelman still contributed a lot and so deserves a Fields medal. Although I cannot imagine that the IMU shares such views, I believe that rewards should not be given in such a way. May be the IMU had found insufficient the opinion of the experts saying that they verified everything and no more doubts are left, and supposed that for such a landmark in the history of mathematics it is wise to wait some more time before announcing it loudly (although for the other Fields medals waiting 3-4 years is sufficient) ? Then there is no need to rush and give an award with a questionable formulation (if some doubts about his proofs are still left) to which Perelman reasonably answered *"It was completely irrelevant for me. Everybody understood that if the proof is correct, then no other recognition is needed."*

Although the Fields medals are often compared with the Nobel prizes, a comparison may concern only their fame, but not the amount of money, which is rather symbolical in the case of Fields prizes. Some say that this is the highest award in mathematics, but in fact it has a flavor of supporting younger mathematicians, which should be below 40 (so, the actual research should be done under the age 37–38). So, it happens that the greatest mathematical achievements cannot receive this prize because of this restriction, the most obvious example is the proof by

A.Wiles of the "Fermat Theorem". An attempt to give a Fields medal to Perelman may be an attempt to support the prestige of the prize.

An award which looks more serious is the *Millennium prize* announced by the *Clay Mathematics Institute* (CMI). By its rules, this prize will be offered two years after a journal publication of the proof. Since Perelman refused to publish his work in a journal, CMI found a solution, which sounds a bit like an anecdote: nobody said that publication should be by the author of the result himself, so one can use for instance the publication of Cao-Zhu to award Perelman the Millennium prize !

The value of this prize, 1 million dollars, may look descent although not that impressive for such a fundamental discovery (still not comparable with the regular top awards in some less serious fields, such as sport and show business). Anyway, the smell of money raised a noise in the mass media and heated passions of people remote from the subject. Some American newspapers insist that Perelman should share this money with Hamilton. In turn, they picture Perelman as a crazy person, repeat various stupid gossips about him, and in particular convinced the public that, in addition to the Fields medal, he already rejected the Millennium prize (which he never did, as is clear from the report of the president of IMU in ICM-2006 and from the interview in the "New Yorker"; besides, this prize was not offered to Perelman yet).

The ideas about sharing the prize began to circulate among mathematicians too, so I would like to make a remark. Giving the Millennium prize after several years of advertising, in the eyes of many people would look as an "official" and final recognition of the merits. If a prize for "solution of the Poincare Conjecture" is shared between two people, it will be understood by many that there are two authors of the solution. This would be completely misleading.

Let me give J.Morgan's comment from his interview in ICM-2006:

Can we say that Perelman has proven the Poincaré Conjecture ?

Yes, I would say and I will say today that Perelman has proven the Poincare Conjecture. But one has to understand that he would not have done it without Hamilton's work. Yet, in the culture of mathematics it is my view that the credit for proving the Poincaré conjecture should go to Perelman.

Like Morgan, I support honoring Hamilton for proposing the Ricci flow project and advancing in it. But Hamilton is not the only one, without whose works Perelman would not have done his research. It is not fair to forget for example A.D.Aleksandrov. And recognizing merits of all the related mathematicians we should first of all respect Perelman by not imposing on him extra coauthors. Otherwise mathematical awards could become just a tool of manipulation by the public's opinion. In such a case, it would be better not to offer any award at all.

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